Learning from the Past for the Future of Mathematics Education :

the SLECC Experimentation

(Savoir Lire Écrire Compter Calculer)

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SLECC (or the 3 Rs) is an educational project conceived and supervised by the ***G****roupe de* ***R****éflexion* ***I****nterdisciplinaire sur les* ***P****rogrammes* (*Interdisciplinary Group for the Study of School Standards*): its current president is the mathematician Jean-Pierre Demailly, now in China for a congress, and I have the charge to stand for him. This project is now experimented mostly in primary schools with the authorization of DGESCO (School department of the ministry of education).

The SLECC program includes a drastic reform of mathematics education for primary school, but viewing it from just a mathematical perspective would greatly reduce its scope and value. In fact, its sphere of activity covers all other disciplines, in particular their relation to mathematics : links with scientific activities are certainly involved, but the main concern is to develop links between language and mathematics from the very beginning of learning, i.e. in France from the G*rande Section de Maternelle (*5/6 years old pupils).

The foundation of the SLECC curriculum is the exact opposite of a general trend in pedagogy introduced at the end of the 60's *which has led to sacrifice the contents of the primary school curriculum under the fallacious principle that a pupil learns better if there is less to learn.* This principle might unfortunately be true for the now common practice of teaching merely mechanisms, procedures and skills of immediate use, namely contents focused on memorizing rather than understanding. However, it turns out to be completely wrong as soon as one aims to teach concepts which are logically interrelated because, in that case, the elimination of certain links in the logical chain of knowledge leaves the remaining notions more difficult or even impossible to learn.

In such a short amount of time, only a small number of subjects can be discussed. We will first investigate on one example the historical evolution which led to the decay of academic programs and to an unfortunate tendency of teaching sparse and disorganized items : namely, the introduction to reading and counting at the end of nursery school (pupils aged 5-6) and in Grade 1 (pupils aged 6-7). We will then discuss three important issues concerning highel levels of primary school.

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[1] Lisbon, December 17th, 2007. "The future of Mathematics Education in Europe" http://www.fmee2007.org/

I) First grades' curriculum :Writing and Division in kindergarten !

Prior to the XIX century a mechanistic and fragmented approach was the accepted way of learning reading and writing. Firstly the alphabet was taught; then syllables, followed by whole words in latin – so that the words couldn't be guessed from their meaning. The next step was to learn words in French and finally sentences were read out aloud in both latin and in French. It was only after this step that pupils could begin to learn how to write (again firstly letters then syllables, etc…) and, of course only if the pupil could afford to pay for the lessons.

At this time most pupils who learnt reading and writing didn't learn arithmetic because again it had to be paid for. Arithmetic was not taught until the pupil had succeeded in learning how to read and write. First of all they learnt how to count, followed by the four operations, which were taught in succession not at the same time.

In the mid-19th. Century the global revolution in education put an end to this mechanistic and fragmented approach to learning. One of the best advocates of this new global revolution was Ferdinand Buisson. (1841-1932). He was the director of elementary teaching at the Ministry Public Instruction between 1872 and 1896 and the author of the monumental *Dictionnaire de pédagogie et d'instruction primaire*[i] (7000 pages in 4 volumes) intended as a reference work for teachers and written by the intellectual elite of the time. Based on intuition and called *méthode intuitive*[ii], the great strength of this revolution, as compared with the preceding methods, which F. Buisson justly describes as archaic, scholastic and medieval, comes from relying on :

- i) the simultaneous teaching of reading and writing : a "phonics" method called "writing-reading"

-ii) the simultaneous learning of counting and calculating[2] or more precisely the simultaneous learning of the 4 operations as counting progresses (*Intuitive calculation*[iii]) whereas the preceding methods first taught counting and then successively each operation separately: in fact,

- the decimal place value system ties counting and computing : 340 does mean 3 *times* 100 *plus* 4 *times* 10
- each operation is defined in relation to the others
- the "intimate knowledge of numbers" (*René Thom*) does not come about unless a number is conceived as the result of various operations : one does not really know 6, once its place in the

[2] I will on purpose omit the central role occupied in the "méthode intuitive" by the simultaneous teaching of not only numbering and counting, but also of mental calculations and of the use of fractions - already in the last year of kindergarten !

counting sequence (between 5 and 7) is established, but one starts to capture its profound meaning by knowing the results of 4+2, 5+1, 7-1, 8-2, 2×3, 6×1, the quotient by 2 of 12 and 13 …

But let us hear what F. Buisson has to say :

> " *Apart from the psychological considerations which inspired it, [the method of intuitive calculation] lets children, on their own and by intuition, perform the essential operations of elementary computation; its aim is to make them familiar with numbers: being familiar with an object means not just knowing its name, it means having seen in all its forms, in all its states, in its diverse relations with other objects; it means being able to compare it with others, to follow it in its transformations, to grasp and measure it, to compose and decompose it at will. Thus treating numbers like any other objects to be presented to the child's intelligence, Grube strongly opposes the old custom of successively teaching the pupils first addition, then substraction, then the other two operations*." [F. Buisson, Calcul intuitif]

Another advantage of this viewpoint is its role in problem solving. Solving arithmetical problems amounts ultimately to look for which type of operation to use so as to reach the desired answer, and a simultaneous learning of all 4 operations allows an early and complete training to such tasks.

This curriculum was applied with very little changes from 1880 to 1970 in last year of Kindergarten and grade 1. Moreover, as is easily shown by the following two examples, the modernity represented by many so called discoveries of pedagogy since 1970 is a return to scholastic methods which rely excessively on memory and rote learning to the detriment of intelligence and intuition :

- i) functional or balanced methods once again separate writing from reading as pupils systematically "read" items they do not know how to write.

- ii) there has been a progressive return to the separation of learning how to count and how to compute since the modern math curriculum which mentions only addition in grade 1. APMEP clearly wrote in 1972 for the same grade : « *One cannot study each natural number as a sum or product of natural numbers, study its "decompositions", because these notions as well as those of difference or quotient will be treated in different steps*. » (1972[iv])

The opposite viewpoint is precisely what characterizes the present SLECC's recommendations for the 3R's in kindergarten and grade 1 :

i) the simultaneous teaching of reading and writing : the "phonics" method called "writing-reading" disqualifies from scratch the decades-long debate of *"whole language reading"*, simply because there cannot be any *"whole language writing"* !

ii) the simultaneous teaching of counting and calculating - or more precisely the simultaneous teaching of the 4 arithmetic operations along with numbering. By contrast, the now common practice of delaying the introduction of subtractions, multiplications and divisions produces this awkward result : during a period of time which can be estimated to 2-3 years after the last year of kindergarten, pupils who are supposed to learn problem solving only have to operate through additions and ... additions. It is no surprise that pupils' brains will become hardwired with addition as the only option ...

II) Three important points of the SLECC's mathematics fundamentals in primary school

1) *The main point of rupture in 1970 : the elimination of magnitudes*[3]

This is not a personal analysis made *a posteriori*, but a declaration published in 1972, by the APMEP, the principal association of teachers of mathematics and the hub of action for the "New Math" reform, in their special issue dedicated to the Bulletin Officiel of January 1970, which had introduced this New Math in elementary school. It is therefore a central and definitive statement:

> *The elimination of "operations with magnitudes " is the truly fundamental mutation brought in by the transitional curricula, the one which profoundly transforms the thought processes in elementary teaching.*

This elimination of operations with magnitudes -- and hence with concrete numbers -- is not argued on its merits by the B.O. (official administration journal), but appears in the following form:

> *Sentences like* 8 apples + 7 apples = 15 apples *are not part of the language of mathematics*

This is of course a pedagogical absurdity, but above all a mathematical one, since in 1968, i.e. two years before the publication of the B.O. and four years prior to the commentary by the APMEP, the great geometer *Hassler Whitney* had published an article providing an axiomatic, "modern" mathematical framework for operations with magnitudes, entitled *The Mathematics of Physical Quantities*. In it, he explicitly says -- and shows by giving an underlying mathematical structure (rays and birays)[4] -- that it is entirely "mathematical" to write :

> 5 cakes + 2 cakes = (5+2) cakes = 7 cakes
>
> or, for instance,
>
> 2 yd = 2 (3 ft) = 6 ft

The context of his introduction makes it clear that he is explicitly taking aim at the New Math, notably by pointing out the absurdity of imposing stilted language like "*Complete:* 2 cm *measure the same as* … mm; 80 mm *measure the same as* … cm*.*", when he says:

> *The fact that* "2 yd" *and* "6 ft" *name the same element of the model enables us to say they are equal; there is no need for such mysterious phrases as* "2 yd measures the same as 6 ft."

[3] Additional informations in *A propos des nombres concrets et abstraits : Un témoignage historique sur l'école primaire française*, Banff , december 2004. http://michel.delord.free.fr/banff.pdf

[4] A more formal way of treating that question from the viewpoint of advanced mathematics would be to invoke tensor calculus.

This proves among other things that the practice of calculating with magnitudes is rather more "modern" than the reduction of all calculation to that on pure numbers.

Another excerpt from the same article of the APMEP journal 1972 issue also describes what we consider to be a denying of the fecund link between the pedagogy of math and physics. It also reveals the origins of the current trends which led to teach mathematics without any intuition from physics, and a "purely experimental" approach of physics totally disconnected with mathematics

> *Indeed, natural numbers are no longer linked to measuring objects from the physical world, and, above all, the operations on these numbers are no longer based on operations on "physical quantities" of our sensible world such as length, weight, price, volumes ...*

An even more drastic consequence is that the rejection of units in operations, in the very first examples where quantities (i.e. numbers with units) are to be calculated along with pure numbers, makes it impossible to grasp the basis of dimensional analysis. We will consider an example presented by Michèle Artigue in 1982, which illustrates the rather wicked character of curricula insisting on teaching orders of magnitude, while disconnecting them from operations on physical quantities and their units:

> *We purposefully gave "idiotic" problems to students. The team of the IREM of Grenoble went even further in breaking the didactical contract by asking elementary school students nonsensical questions such as: "In a class, these are 4 rows of 8 seats, how old is the teacher?"; and we were shocked to observe that most of the elementary school students made an effort to solve these problems as though nothing was wrong ; and they did not choose the mathematical operations at random: the teacher was determined to be 32 years old.*

In my opinion there is nothing shocking about the procedure followed by the student[10] who only acted as he was taught in accordance with the official curriculum for the past 30 years. This situation arises when:

- the student was taught only pure numbers, and therefore was not given the definition of the operations (which is possible only -- as was done in the chapter on the "meaning of operations" -- in the context of magnitudes), hence had no criteria by which to select operations or subsequently verify that the result made sense in terms of dimensions
- in addition, the heavy insistence on calculating orders of magnitude leaves him with this calculation as the only guide to his choice of operations,

The student proceeds in the following manner: he calculates 8+4=12, 8-4=4, 8x4=32, 8/4=2, 4/8=0.5, and since, in terms of order of magnitude, the teacher cannot possibly be 12, 4, 2 or 0.5 years old, her age must be 32 years.

Not having a definition of multiplication, the student cannot know that any multiplication can be expressed in terms of multiplicand, a concrete number with a specific dimensional unit, and multiplier indicating the number of repetitions of the multiplicand, product having the same unit as the multiplicand.

If he had known this, he would have been able to choose either the number of rows or the number of seats as the multiplicand, but he would have noticed that in either case the result of the multiplication could not have been a number of years. Even if he had known only the rule (a procedural rule which is extremely important at the beginning of teaching) *always write the multiplicand with its unit first: if you want meters in a multiplication, start with meters*, he would not even have started writing the multiplication, since in writing down the multiplicand (either seats or rows), he would have known that he could not get years as the product.

It would be a little tedious to develop the rules of dimensional calculation for each operation in relation with their definitions (and the appropriate and effective ways of expressing these for each level of teaching). We will choose the common definition of multiplication corresponding to *3u* × *5* = *15u* in a 1912 book[v] noting that all of the "other multiplications" (for example 3€/m × 5m= 15€, 3m × 5m = 15m^2, 3m^2 × 5m = 15m^3…) should first be taught in this format.

Multiplication

Meaning of the operation

68. **Multiplication** is an operation whereby one repeats a number called **multiplicand**, a number of times indicated by another number called **multiplier**. The result is called **product**. […][vi]

70. The multiplicand and the multiplier are called **factors** of the product.

71. Multiplication is indicated by the sign × (**multiplied by)** which is written between the numbers to be multiplied: 8×5 (8 *multiplied by* 5).

72. Multiplication is only an *abbreviation of addition*.

73. The *multiplicand* is always a *concrete* number, that is one which describes a specific object, such as trees, meters, dollars, …

74. The *multiplier* is an *abstract* number that indicates only the number of times that one repeats the multiplicand.

75. The *product* is always in **units** *similar* to those in the multiplicand.

Technicality of the operation

76. Multiplicand and multiplier have one digit...

77. Multiplicand has two digits and multiplier one…

2) Why should one teach *pencil and paper algorithms, specially for division ?* [5]

> "*We would like to emphasize that the standard algorithms of arithmetic are more than just 'ways to get the answer' -- that is, they have theoretical as well as practical significance. For one thing, all the algorithms of arithmetic are preparatory for algebra, since there are (again, not by accident, but by virtue of the construction of the decimal system) strong analogies between arithmetic of ordinary numbers and arithmetic of polynomials.*"
> Notices of the AMS, February 1998

Let us first mention that the ideology which denies any interest to teaching pencil and paper standard algorithms has for a long time been dominant in France as well as internationally, even though it may now appear as less influential :

- in France, in 1984, the COPREM (an official board of the Ministry of Education) was explaining :

> « *An excellent performance in practicing pencil and paper algorithms is no longer an intrinsic necessity nowadays, since eventually an electronic calculator can serve as a "computing prosthesis". It is therefore no longer important to reach a reliable execution of operations made by hand: in case of emergency, one could very well buy a calculator for a very modest amount of money (a few cigarette packages) at the nearest shop* »[vii]

- in United States, Steven Leinwand, member of the panel set up by M.Riley, state secretary of Education in 1999, in charge of expertizing the curricula, was also writing :

> *"It's time to recognize that, for many students, real mathematical power, on the one hand, and facility with multidigit, pencil-and-paper computational algorithms, on the other, are mutually exclusive. In fact, it's time to acknowledge that continuing to teach these skills to our students is not only unnecessary, but counterproductive and downright dangerous.*".
> S. Leinwand "*It's Time To Abandon Computational Algorithms,*" February 9, 1994, Education Week on the Web

A first justification of the necessity of teaching operation algorithms is quite simple: one can understand only things which one is able to do concretely. A pupil cannot understand an operation if he cannot follow all steps which lead him to the result, starting from two initial numbers written explicitly in some numeration basis, say 10. For this, it is necessary that the list of cases he has to treat are sufficiently different and general to lead him to be confident that the techniques he knows will be applicable to arbitrary numbers. In that sense, using a calculator to perform operations that pupils do not master at all, *as is still*

[5] More arguments in *Pourquoi apprendre à faire les opérations à la main ?* Lille, 29 septembre 2006 - http://michel.delord.free.fr/lille-29092006.pdf

expressly recommended in the French programs published in 2002, for instance for the quotient of a decimal number by a whole number, not only is extremely poor on the ground of acquiring knowledge, but also induces a completely inappropriate confidence in the black magic of electronic devices. Now we will answer our initial question about the technically most complicated operation, namely division. **Why should one learn the *paper and pencil* algorithm of division ?**

A) Mastering the algorithm of divisions is the best training for the three other operations and for mental calculations.

B) Practicing divisions without writing explicitly subtractions, but just memorizing them - which has been a rather specific French tradition in schools - has shown to be one of the best exercises to acquire techniques of mental calculations as well as to consolidate the knowledge of operation tables.

C) A knowledge of basic properties of division such as «*If one divides -or multiplies- the dividend and the divisor by the same number, the quotient does not change and the remainder is divided -or multiplied- by this number* » is an essential introduction

- to the understanding of the concept of fraction and especially their simplification
- to the logical understanding of the algorithm used for division of decimal numbers : actually, to divide 2.732 by 0.17, one replaces this division, after multiplying both dividend and divisor by 100, by the division of 273.2 by 17, which will have the same quotient, but the remainder of which will be multiplied by 100.

D) A knowledge of the algorithme of division is the only way to grasp the concept of rational numbers and differentiate them from decimal numbers (a differentiation which is not permitted by numerical calculators, even very sophisticated ones). In fact, the only way to check that a fraction represents a rational number is to verify that non only the quotient appears to have a repetition of identical sequences of digits, but also that the sequence of remainders also produces repeated numbers, a fact which can be observed immediately while performing a pencil and paper division.

E) The approximation of rational number by the sequence of the decimal intermediate quotients is the occasion of a first contact, already at the end of primary school, with the deeper mathematical concepts of limits and sequential limits.

The sequence 0.7 ; 0.71 ; 0.714 ; 0.7142 … obtained by taking the division 5/7 actually has 5/7 as a limit because its terms are by *construction the decimal approximations of the quotient provided by the algorithm.*

F) From the viewpoint of algorithmics and computer science, the algorithm of pencil and paper division is probably one of the earliest *non trivial* algorithms employed by humans.

G) Learning the algorithm of division on whole numbers (along with the other operations) appears to be an excellent preparation to the similar algorithm which will be taught later for dividing polynomials.

Indeed, in very much the same way that the quotient of 123 by 11 is 11 and the remainder is 2, the quotient of $1X^2 + 2X + 3$ by $1X + 1$ is $1X + 1$ and the remainder is 2.[viii]

3) Technicality against meaning : a bogus concept of pedagogy

It is absurd to oppose the meaning of an operation and its technique, since a course involving an approach of dimensional analysis can derive the technique from the meaning. I will take only one example, namely addition, but the idea is valid as well for all arithmetic operations, and also for the teaching of the native language for which an intimate knowledge of the rules of grammar is a necessary condition for a deeper understanding of the language itself. The teachers of my era, in the 1950's, would have been quite familiar with the following precept : « *One should not add cows and pigs* » or «*One should not add cloths and towels* ». However, it is maybe preferable to be more explicit and say, as soon as the sentence becomes understandable to pupils «*One can only add physical quantities of the same kind, and one can perform the operations only after they are expressed in the same units* ». It is even certainly commendable to take examples using the *International System of Units so* as to make pupils familiar with them.

i) one cannot add three meters and two liters because they are not quantities of the same nature

ii) one can add three meters and two decimeters but, in order to find the result, one does not add three and two though, because the quantities are indeed of the same nature - lengths - but they are not expressed with the same units

iii) in order to add three meters and two decimeters, one replaces three meters by thirty decimeters and one finds 3 m + 2 dm = 32 dm

Consider the addition of 2213 and 473, and that of 2.213 and 47.3

		4	7	3
+	2	2	1	3
	2	6	8	6

If one adds 4 and 2, it is indeed because 4 and 2 express quantities of *same nature*, namely hundreds.

	4	7	.	3		
+		2	.	2	1	3
	4	9	.	5	1	3

One adds 3 and 2 because they are both tenths, thus quantities of the same nature, which is a justification of the practical rule to follow : *align decimal points.*

This is essentially the same definition which allows to introduce efficiently the addition of fractions, and this is indeed what the Dictionnaire Pédagogique of 1882 was already suggesting, except maybe for the exact wording of the rule : «*The addition of fractions (or of fractional expressions) suppose that they have the same denominator, because one can only add quantities of the same kind expressed with the same denomination* ».[ix]

i See some extracts at : http://michel.delord.free.fr/dp.html

ii Ferdinand Buisson, Article *Intuition et méthode intuitive* in *Dictionnaire de pédagogie et d'instruction primaire*, Hachette, 1887. Tome 2 de la première partie, pages 1374 à 1377. http://michel.delord.free.fr/fb_intuit.pdf

iii Ferdinand Buisson, *Calcul Intuitif* http://michel.delord.free.fr/fb-calcintuit.pdf

iv Marguerite Robert, *Réflexions sur le programme rénové : Un nouvel état d'esprit,* Pages 15 à 58. Extrait de *La mathématique à l'école élémentaire*, Paris, Supplément au bulletin APMEP n° 282, 1972, 502 pages).

v *Brouet et Haudricourt Frères,* Arithmétique et système métrique Cours Moyen, *Librairies-Imprimeries réunies, Paris, 1912*

vi Item 69 gives definition of multiplication based on proportionality:

> *Multiplication can also be defined as follows:*
>
> *69. – Multiplication is an operation whose aim is to find a number called the product, which is to the multiplicand as the multiplier is to unity.*

vii "*Contribution à l'enseignement mathématique contemporain : Analyse des contenus, méthodes, progressions, relatifs aux principaux thèmes des programmes : La proportionnalité / Le calcul numérique*" MEN CRDP Strasbourg Dépôt légal 1987. Responsables de la rédaction de ce texte : la direction des collèges, des lycées et l'inspection générale de mathématiques.
Compléments : http://michel.delord.free.fr/txt1999/1_opinions.html

viii Read Michel Delord : *Opérations arithmétiques et algèbre des polynômes ou Apprend-on seulement les opérations pour* trouver le résultat *?* http://michel.delord.free.fr/ar-alg.pdf

ix *Henri Sonnet*, Article *Fractions* de la Partie II du *Dictionnaire Pédagogique*, pages 792 à 798.